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# Jointly Stable Matchings\*

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## Abstract

In the stable marriage problem, we are given a set of men, a set of women, and each person's preference list. Our task is to find a stable matching, that is, a matching admitting no unmatched (man, woman)-pair each of which improves the situation by being matched together. It is known that any instance admits at least one stable matching. In this paper, we consider a natural extension where  $k(\geq 2)$  sets of preference lists  $L_i$  ( $1 \leq i \leq k$ ) over the same set of people are given, and the aim is to find a *jointly stable matching*, a matching that is stable with respect to all  $L_i$ . We show that the decision problem is NP-complete already for  $k = 2$ , even if each person's preference list is of length at most four, while it is solvable in linear time for any  $k$  if each man's preference list is of length at most two (women's lists can be of unbounded length). We also show that if each woman's preference lists are same in all  $L_i$ , then the problem can be solved in linear time.

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## 1 Introduction

In this paper, we focus on the *stable marriage problem* [3] with incomplete preference lists (SMI). An instance  $I$  of SMI is a triple  $I = (U, W, L)$ , where  $U$  and  $W$  are the sets of men and women, respectively, such that  $|U| = |W| (= n)$ , and  $L$  is the set of  $2n$  *preference lists*, one for each person. A person  $p$ 's preference list in  $L$  is denoted by  $L(p)$ . Each person's preference list strictly orders a subset of the members of the opposite gender. If a person  $p$  is included in  $L(q)$ , we say that  $p$  is *acceptable* to  $q$ . If  $p$  is acceptable to  $q$  and vice versa,  $(p, q)$  is called an *acceptable pair*.

A *matching* is a set of acceptable (man, woman)-pairs in which no person appears more than once. For a matching  $M$ , a man  $m$ , and a woman  $w$ , if  $(m, w) \in M$  then we write  $M(m) = w$  and  $M(w) = m$ . If there is no  $w$  (respectively,  $m$ ) such that  $(m, w) \in M$ , we say that  $m$  (respectively,  $w$ ) is *single* or *unmatched* in  $M$ . For a matching  $M$ , if (i)  $(m, w)$  is an acceptable pair, (ii)  $m$  is single in  $M$  or prefers  $w$  to  $M(m)$ , and (iii)  $w$  is single in  $M$  or prefers  $m$  to  $M(w)$ , then we say that  $(m, w)$  is a *blocking pair* for  $M$  in  $L$ , or  $(m, w)$  *blocks*  $M$  in  $L$ . If there is no blocking pair for  $M$  in  $L$ , then we say that  $M$  is *stable* in  $L$ . It is well-known that any SMI instance admits at least one stable matching [3].

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In this paper, we consider an extension of SMI where two or more sets of preference lists are given. An instance  $I$  of the *Stable Marriage problem with  $k$  Incomplete lists (SM $k$ I)* is a  $(k+2)$ -tuple  $I = (U, W, L_1, L_2, \dots, L_k)$ , where  $U$  and  $W$  are the same as above, and each  $L_i$  is a set of preference lists. It asks if there exists a matching  $M$  that is stable in every  $L_i$ . Such a matching  $M$  is called *jointly stable*. Let  $a$  and  $b$  be positive integers. The restriction of SM $k$ I where the lengths of preference lists of men are at most  $a$  and those of women are at most  $b$  is denoted by  $(a, b)$ -SM $k$ I. If  $a$  (respectively,  $b$ ) is  $\infty$ , it means that the lengths of men's (respectively, women's) preference lists are unbounded. Surprisingly, although this problem is a natural extension of the classical stable marriage problem, to the best of the authors' knowledge it has not been considered before in the literature. Note that since the number of stable matchings grows exponentially in the size of the input [7, 5, 13], an algorithm of enumerating all the stable matchings for each  $L_i$  and computing their intersection is not polynomial-time bounded.

Besides its theoretical interest, the problem has several applications: Consider a scenario of assigning medical residents to hospitals, where each resident needs to take training in three fixed clinical departments, e.g., surgery, pediatrics, and internal medicine, at an assigned hospital. A resident  $r$  ranks hospitals according to her preference, but her ranking of hospitals may differ depending on clinical departments. As a result, she has three (possibly different) preference lists over hospitals,  $L_1(r)$  for surgery,  $L_2(r)$  for pediatrics, and  $L_3(r)$  for internal medicine. On the other hand, each clinical department may have its own criteria for ranking residents, so each hospital  $h$  has three independent preference lists over residents,  $L_1(h)$  from surgery,  $L_2(h)$  from pediatrics, and  $L_3(h)$  from internal medicine. Clearly a blocking pair in some  $L_i$  may cause dissatisfaction to the corresponding resident and department, so we want to avoid such an assignment. Another example is a match making of Judo team competition. Suppose that there are five different weight classes, and one team consists of five players, each from each class. As a personal preference, a player  $p$  of team  $T$  who belongs to the weight class  $C$  is interested in only the players of the same class  $C$ , who are potential candidates for  $p$ 's opponent. Therefore, each team has five preference lists corresponding to weight classes, and a matching avoiding blocking pairs in any class is desirable. Precisely speaking, the first and the second examples may be suitable to the Hospitals/Residents and the stable roommates, respectively, but we consider in this paper the stable marriage model as a first step.

## 1.1 Our Results

We show that  $(4, 4)$ -SM $k$ I is NP-complete for any  $k \geq 2$ , while  $(2, \infty)$ -SM $k$ I is solvable in time  $O(kn)$  for any  $k$ . Therefore the complexity of  $(3, \ell)$ -SM $k$ I for  $\ell \geq 3$  is left open.

We also show that SM $k$ I (with unbounded-length preference lists) is solvable in polynomial time if  $L_1(w) = L_2(w) = \dots = L_k(w)$  holds for every woman  $w$ . This can be thought of as a case where each woman has only one preference list, and one of its interpretations is a modification of the previous example of assigning residents to hospitals, where each resident has three preference lists as above, but each hospital has one preference list determined by e.g., a personnel director of the hospital, rather than three independent lists coming from each clinical department.

## 1.2 Related Work

As noted above, there seems to be no research on stable matching problems considering multiple preference lists over the same set of people. Only the related work we have found is the *bistable matching problem* introduced by Weems [14]; given an instance  $I$  of the stable

marriage problem (where preference lists are complete), let  $\hat{I}$  be the instance obtained by reversing the ordering of each preference list of  $I$ . A matching is *bistable* if it is stable in both  $I$  and  $\hat{I}$ . This is a special case of SM2I where all the preference lists are complete and  $L_1(p)$  is a reversed order of  $L_2(p)$  for every person  $p$ . Weems showed an  $O(n^2)$ -time algorithm to find a bistable matching or to report that none exists. Sethuraman and Teo [11] showed that the bistable roommates problem can also be solved in polynomial time. See pages 293–296 of [10] for a brief survey.

## 2 NP-completeness

In this section, we show the hardness result.

► **Theorem 1.** *For  $k \geq 2$ ,  $(4, 4)$ -SM $k$ I is NP-complete.*

**Proof.** It is easy to see that  $(4, 4)$ -SM $k$ I is in NP. In the following, we show that  $(4, 4)$ -SM2I is NP-hard. To show the NP-hardness for general  $k$ , one may simply set  $L_2 = L_3 = \dots = L_k$  in the reduction.

We give a polynomial-time reduction from the well-known NP-complete problem 3CNF SAT. The definition of 3CNF SAT is as follows. Let  $x$  be a binary variable that takes 1(true) or 0(false). A *literal* is a variable  $x$  or its negation  $\bar{x}$ . A *clause* is a disjunction of literals, and a *Conjunctive Normal Form (CNF) formula* is a conjunction of clauses. A *3CNF formula* is a CNF formula in which each clause contains at most three literals. An instance of 3CNF SAT is a 3CNF formula  $f$  and it asks if there exists an assignment to variables that makes  $f$  true. We may assume without loss of generality that each clause contains *exactly* three literals. (If a clause contains less than three literals, then repeat the same literal.)

Let  $f$  be an instance of 3CNF SAT, with variables  $x_1, x_2, \dots, x_n$  and clauses  $C_1, C_2, \dots, C_m$ . We construct an instance  $I$  of  $(4, 4)$ -SM2I. For each  $i$  ( $1 \leq i \leq n$ ), let  $s_i$  be the number of occurrences of the variable  $x_i$ . For the  $j$ th literal of the variable  $x_i$  ( $1 \leq j \leq s_i$ ), we introduce two men  $a_{i,j}$  and  $b_{i,j}$  and two women  $c_{i,j}$  and  $d_{i,j}$ . We call them *literal men* and *literal women*. For each clause  $C_\ell$ , we introduce nine men  $u_\ell^i$  ( $1 \leq i \leq 9$ ) and nine women  $v_\ell^i$  ( $1 \leq i \leq 9$ ). We call them *clause men* and *clause women*. Note that there are  $15m$  men and  $15m$  women in total.

The preference lists of literal people and clause people are given in Figures 1 and 2, respectively. In  $a_{i,1}$  and  $d_{i,1}$ 's preference lists of  $L_2$  in Fig. 1,  $c_{i,j-1}$  and  $b_{i,j-1}$  are null; hence their preference lists are of length two. Similarly, in  $b_{i,s_i}$  and  $c_{i,s_i}$ 's preference lists of  $L_2$ ,  $d_{i,j+1}$  and  $a_{i,j+1}$  are null. We then explain  $U_{i,j}$  and  $V_{i,j}$  in Fig. 1. Suppose that the  $j$ th occurrence of  $x_i$  is the  $t$ th literal of the clause  $C_\ell$ . If this literal is positive, then  $U_{i,j}$  is null and  $V_{i,j} = v_\ell^4$  if  $t = 1$ ,  $V_{i,j} = v_\ell^7$  if  $t = 2$ , and  $V_{i,j} = v_\ell^1$  if  $t = 3$ . If it is negative, then  $V_{i,j}$  is null and  $U_{i,j} = u_\ell^1$  if  $t = 1$ ,  $U_{i,j} = u_\ell^4$  if  $t = 2$ , and  $U_{i,j} = u_\ell^7$  if  $t = 3$ . Finally, we explain  $B_{\ell,1}, B_{\ell,2}, B_{\ell,3}, D_{\ell,1}, D_{\ell,2}$ , and  $D_{\ell,3}$  in Fig. 2. Suppose that, for  $t = 1, 2, 3$ , the  $t$ th literal of the clause  $C_\ell$  is the  $j$ th occurrence of  $x_i$ . If this literal is positive, then  $D_{\ell,t}$  is null and  $B_{\ell,t} = b_{i,j}$ ; otherwise,  $B_{\ell,t}$  is null and  $D_{\ell,t} = d_{i,j}$ . Now the reduction is completed. It is not hard to see that the reduction can be performed in polynomial time and each person's preference list is of length at most four.

$L_1$	$a_{i,j}:$	$c_{i,j}$		$d_{i,j}$		$c_{i,j}:$	$b_{i,j}$		$a_{i,j}$
	$b_{i,j}:$	$d_{i,j}$	$V_{i,j}$	$c_{i,j}$		$d_{i,j}:$	$a_{i,j}$	$U_{i,j}$	$b_{i,j}$
$L_2$	$a_{i,j}:$	$c_{i,j}$	$c_{i,j-1}$	$d_{i,j}$		$c_{i,j}:$	$b_{i,j}$	$a_{i,j+1}$	$a_{i,j}$
	$b_{i,j}:$	$d_{i,j}$	$d_{i,j+1}$	$c_{i,j}$		$d_{i,j}:$	$a_{i,j}$	$b_{i,j-1}$	$b_{i,j}$

■ **Figure 1** Preference lists of literal people corresponding to the  $j$ th occurrence of the variable  $x_i$  ( $1 \leq j \leq s_i$ ).

$L_1$	$u_\ell^1:$	$v_\ell^1$	$v_\ell^2$	$D_{\ell,1}$	$v_\ell^3$		$v_\ell^1:$	$u_\ell^2$	$u_\ell^3$	$B_{\ell,3}$	$u_\ell^1$	
	$u_\ell^2:$	$v_\ell^2$	$v_\ell^3$		$v_\ell^1$		$v_\ell^2:$	$u_\ell^3$	$u_\ell^1$		$u_\ell^2$	
	$u_\ell^3:$	$v_\ell^3$	$v_\ell^1$		$v_\ell^2$		$v_\ell^3:$	$u_\ell^1$	$u_\ell^2$		$u_\ell^3$	
	$u_\ell^4:$	$v_\ell^4$	$v_\ell^5$	$D_{\ell,2}$	$v_\ell^6$		$v_\ell^4:$	$u_\ell^5$	$u_\ell^6$	$B_{\ell,1}$	$u_\ell^4$	
	$u_\ell^5:$	$v_\ell^5$	$v_\ell^6$		$v_\ell^4$		$v_\ell^5:$	$u_\ell^6$	$u_\ell^4$		$u_\ell^5$	
	$u_\ell^6:$	$v_\ell^6$	$v_\ell^4$		$v_\ell^5$		$v_\ell^6:$	$u_\ell^4$	$u_\ell^5$		$u_\ell^6$	
	$u_\ell^7:$	$v_\ell^7$	$v_\ell^8$	$D_{\ell,3}$	$v_\ell^9$		$v_\ell^7:$	$u_\ell^8$	$u_\ell^9$	$B_{\ell,2}$	$u_\ell^7$	
	$u_\ell^8:$	$v_\ell^8$	$v_\ell^9$		$v_\ell^7$		$v_\ell^8:$	$u_\ell^9$	$u_\ell^7$		$u_\ell^8$	
	$u_\ell^9:$	$v_\ell^9$	$v_\ell^7$		$v_\ell^8$		$v_\ell^9:$	$u_\ell^7$	$u_\ell^8$		$u_\ell^9$	
$L_2$	$u_\ell^1:$	$v_\ell^1$	$v_\ell^4$	$v_\ell^2$		$v_\ell^3$	$v_\ell^1:$	$u_\ell^2$		$u_\ell^3$	$u_\ell^7$	$u_\ell^1$
	$u_\ell^2:$	$v_\ell^2$		$v_\ell^3$	$v_\ell^5$	$v_\ell^1$	$v_\ell^2:$	$u_\ell^3$	$u_\ell^8$	$u_\ell^1$		$u_\ell^2$
	$u_\ell^3:$	$v_\ell^3$		$v_\ell^1$		$v_\ell^2$	$v_\ell^3:$	$u_\ell^1$		$u_\ell^2$		$u_\ell^3$
	$u_\ell^4:$	$v_\ell^5$	$v_\ell^7$	$v_\ell^6$		$v_\ell^4$	$v_\ell^4:$	$u_\ell^4$		$u_\ell^5$	$u_\ell^1$	$u_\ell^6$
	$u_\ell^5:$	$v_\ell^6$		$v_\ell^4$	$v_\ell^8$	$v_\ell^5$	$v_\ell^5:$	$u_\ell^5$	$u_\ell^2$	$u_\ell^6$		$u_\ell^4$
	$u_\ell^6:$	$v_\ell^4$		$v_\ell^5$		$v_\ell^6$	$v_\ell^6:$	$u_\ell^6$		$u_\ell^4$		$u_\ell^5$
	$u_\ell^7:$	$v_\ell^9$	$v_\ell^1$	$v_\ell^7$		$u_\ell^8$	$v_\ell^7:$	$u_\ell^9$		$u_\ell^7$	$u_\ell^4$	$u_\ell^8$
	$u_\ell^8:$	$v_\ell^7$		$v_\ell^8$	$v_\ell^2$	$v_\ell^9$	$v_\ell^8:$	$u_\ell^7$	$u_\ell^5$	$u_\ell^8$		$u_\ell^9$
	$u_\ell^9:$	$v_\ell^8$		$v_\ell^9$		$v_\ell^7$	$v_\ell^9:$	$u_\ell^8$		$u_\ell^9$		$u_\ell^7$

■ **Figure 2** Preference lists of clause people corresponding to the  $\ell$ th clause.

We then proceed to the correctness proof. We first define partial matchings. For each  $i$  and  $j$ , we define  $M_{i,j}^1 = \{(a_{i,j}, c_{i,j}), (b_{i,j}, d_{i,j})\}$  and  $M_{i,j}^0 = \{(a_{i,j}, d_{i,j}), (b_{i,j}, c_{i,j})\}$ . For each  $\ell$ , we define

$$\begin{aligned}
 M_\ell^1 &= \{(u_\ell^1, v_\ell^3), (u_\ell^2, v_\ell^1), (u_\ell^3, v_\ell^2), (u_\ell^4, v_\ell^4), (u_\ell^5, v_\ell^5), (u_\ell^6, v_\ell^6), (u_\ell^7, v_\ell^8), (u_\ell^8, v_\ell^9), (u_\ell^9, v_\ell^7)\}, \\
 M_\ell^2 &= \{(u_\ell^1, v_\ell^2), (u_\ell^2, v_\ell^3), (u_\ell^3, v_\ell^1), (u_\ell^4, v_\ell^6), (u_\ell^5, v_\ell^4), (u_\ell^6, v_\ell^5), (u_\ell^7, v_\ell^7), (u_\ell^8, v_\ell^8), (u_\ell^9, v_\ell^9)\}, \text{ and} \\
 M_\ell^3 &= \{(u_\ell^1, v_\ell^1), (u_\ell^2, v_\ell^2), (u_\ell^3, v_\ell^3), (u_\ell^4, v_\ell^5), (u_\ell^5, v_\ell^6), (u_\ell^6, v_\ell^4), (u_\ell^7, v_\ell^9), (u_\ell^8, v_\ell^7), (u_\ell^9, v_\ell^8)\}.
 \end{aligned}$$

Suppose that  $f$  is satisfiable and let  $T$  be a satisfying assignment. We will construct a jointly stable matching  $M$  for  $I$ . If  $T(x_i) = 1$ , then we let  $M_{i,j}^1 \subseteq M$  for all  $j$ . If  $T(x_i) = 0$ , then we let  $M_{i,j}^0 \subseteq M$  for all  $j$ . Suppose that the clause  $C_\ell$  is satisfied by its  $t$ th literal (if there are more than one true literal, choose one arbitrarily). Then we let  $M_\ell^t \subseteq M$ . We show that  $M$  is jointly stable.

► **Lemma 2.** *The matching  $M$  constructed as above is jointly stable.*

**Proof.** Consider literal people corresponding to  $x_i$ , namely  $a_{i,j}$ ,  $b_{i,j}$ ,  $c_{i,j}$ , and  $d_{i,j}$  ( $1 \leq j \leq s_i$ ). If  $T(x_i) = 1$ , then all the men are matched with their first choices in both  $L_1$  and  $L_2$ . Similarly, if  $T(x_i) = 0$ , then all the women are matched with their first choices. Therefore, no blocking pair arises within literal people corresponding to the same variable. Since literal people corresponding to different variables are unacceptable to each other, no blocking pair occurs between them.

As for the 18 people corresponding to the clause  $C_\ell$ , we can easily verify that, in any of  $M_\ell^1$ ,  $M_\ell^2$ , and  $M_\ell^3$ , no blocking pair arises among them. Also, since clause people corresponding to different clauses are unacceptable to each other, no blocking pair occurs between them.

Finally, we consider a possibility of a blocking pair between a literal person and a clause person. Consider the clause  $C_\ell$ . First, suppose that  $M_\ell^1$  is chosen as a part of  $M$ . By construction of  $M$ , this means that the clause  $C_\ell$  is satisfied by its first literal. Suppose that this literal is the  $j$ th occurrence of  $x_i$ , and that it is a positive literal. Then by construction of preference lists,  $D_{\ell,1}$  is null and  $B_{\ell,1} = b_{i,j}$ , so only the possible blocking pair is  $(b_{i,j}, v_{\ell,4})$  in  $L_1$ . However, since  $C_\ell$  is satisfied by the first literal, it must be the case that  $T(x_i) = 1$ . By construction of  $M$ ,  $M_{i,j}^1 \subseteq M$  and hence  $b_{i,j}$  is matched with his first choice woman in  $L_1$ , so he cannot form a blocking pair. Now suppose that the first literal of  $C_\ell$  is the  $j$ th occurrence of  $x_i$  and it is a negative literal. Then  $B_{\ell,1}$  is null and  $D_{\ell,1} = d_{i,j}$ , so, only the possible blocking pair is  $(u_{\ell,1}, d_{i,j})$  in  $L_1$ . However, since  $C_\ell$  is satisfied by the first literal, we have that  $T(x_i) = 0$  and hence  $d_{i,j}$  is matched with her first choice man in  $L_1$ , so  $d_{i,j}$  cannot form a blocking pair. For the other two cases, that is, the case that  $M_\ell^2$  is chosen and  $M_\ell^3$  is chosen, we can show that there is no blocking pair by a similar argument. ◀

Conversely, suppose that  $I$  admits a jointly stable matching  $M$ . We construct a satisfying assignment  $T$  of  $f$ . First, we see basic properties of  $M$ .

► **Lemma 3.** *For each  $i$ , either  $M_{i,j}^1 \subseteq M$  for all  $j$ , or  $M_{i,j}^0 \subseteq M$  for all  $j$ .*

**Proof.** We first show that, for each  $i$  and  $j$ , either  $M_{i,j}^1 \subseteq M$  or  $M_{i,j}^0 \subseteq M$ . Suppose not. Since  $c_{i,j}$  and  $d_{i,j}$  are the only acceptable men to  $a_{i,j}$  and  $b_{i,j}$  in  $L_1$  and  $L_2$  in common, at least one of  $a_{i,j}$  and  $b_{i,j}$ , say  $m_{i,j}$ , is single in  $M$ . For the same reason, at least one of  $c_{i,j}$  and  $d_{i,j}$ , say  $w_{i,j}$ , is single in  $M$ . Then  $(m_{i,j}, w_{i,j})$  blocks  $M$  (in both  $L_1$  and  $L_2$ ), a contradiction.

Now suppose that the statement of the lemma is false. Then there are  $i$  and  $j$  ( $1 \leq j \leq s_i - 1$ ) such that (i)  $M_{i,j}^1 \subseteq M$  and  $M_{i,j+1}^0 \subseteq M$  or (ii)  $M_{i,j}^0 \subseteq M$  and  $M_{i,j+1}^1 \subseteq M$ . In case of (i),  $(a_{i,j+1}, c_{i,j})$  blocks  $M$  in  $L_2$ , while in case of (ii),  $(b_{i,j}, d_{i,j+1})$  blocks  $M$  in  $L_2$ , a contradiction. ◀

► **Lemma 4.** *For each  $\ell$ , either  $M_\ell^1 \subseteq M$ ,  $M_\ell^2 \subseteq M$ , or  $M_\ell^3 \subseteq M$ .*

**Proof.** Suppose that there is a man  $m_\ell \in \{u_\ell^1, u_\ell^2, u_\ell^3\}$  who is not matched with any of  $v_\ell^1$ ,  $v_\ell^2$ , and  $v_\ell^3$  in  $M$ . Note that  $D_{\ell,1}$  is a literal woman (if not null), who is not acceptable to  $u_\ell^1$  in  $L_2$ . Hence it must be the case that  $m_\ell$  is single in  $M$ . By a similar argument, there

■ **Table 1** 27 matchings and corresponding blocking pairs in  $L_2$ .

Matching	BP		Matching	BP		Matching	BP
$X_\ell^1 \cup Y_\ell^1 \cup Z_\ell^1$	$(u_\ell^7, v_\ell^1)$		$X_\ell^2 \cup Y_\ell^1 \cup Z_\ell^1$	$(u_\ell^5, v_\ell^8)$		$X_\ell^3 \cup Y_\ell^1 \cup Z_\ell^1$	$(u_\ell^5, v_\ell^8)$
$X_\ell^1 \cup Y_\ell^1 \cup Z_\ell^2$	$(u_\ell^7, v_\ell^1)$		$X_\ell^2 \cup Y_\ell^1 \cup Z_\ell^2$	$(u_\ell^8, v_\ell^2)$		$X_\ell^3 \cup Y_\ell^1 \cup Z_\ell^2$	–
$X_\ell^1 \cup Y_\ell^1 \cup Z_\ell^3$	$(u_\ell^4, v_\ell^7)$		$X_\ell^2 \cup Y_\ell^1 \cup Z_\ell^3$	$(u_\ell^5, v_\ell^8)$		$X_\ell^3 \cup Y_\ell^1 \cup Z_\ell^3$	$(u_\ell^5, v_\ell^8)$
$X_\ell^1 \cup Y_\ell^2 \cup Z_\ell^1$	$(u_\ell^7, v_\ell^1)$		$X_\ell^2 \cup Y_\ell^2 \cup Z_\ell^1$	$(u_\ell^1, v_\ell^4)$		$X_\ell^3 \cup Y_\ell^2 \cup Z_\ell^1$	$(u_\ell^1, v_\ell^4)$
$X_\ell^1 \cup Y_\ell^2 \cup Z_\ell^2$	$(u_\ell^7, v_\ell^1)$		$X_\ell^2 \cup Y_\ell^2 \cup Z_\ell^2$	$(u_\ell^1, v_\ell^4)$		$X_\ell^3 \cup Y_\ell^2 \cup Z_\ell^2$	$(u_\ell^1, v_\ell^4)$
$X_\ell^1 \cup Y_\ell^2 \cup Z_\ell^3$	–		$X_\ell^2 \cup Y_\ell^2 \cup Z_\ell^3$	$(u_\ell^1, v_\ell^4)$		$X_\ell^3 \cup Y_\ell^2 \cup Z_\ell^3$	$(u_\ell^1, v_\ell^4)$
$X_\ell^1 \cup Y_\ell^3 \cup Z_\ell^1$	$(u_\ell^7, v_\ell^1)$		$X_\ell^2 \cup Y_\ell^3 \cup Z_\ell^1$	–		$X_\ell^3 \cup Y_\ell^3 \cup Z_\ell^1$	$(u_\ell^2, v_\ell^5)$
$X_\ell^1 \cup Y_\ell^3 \cup Z_\ell^2$	$(u_\ell^7, v_\ell^1)$		$X_\ell^2 \cup Y_\ell^3 \cup Z_\ell^2$	$(u_\ell^8, v_\ell^2)$		$X_\ell^3 \cup Y_\ell^3 \cup Z_\ell^2$	$(u_\ell^2, v_\ell^5)$
$X_\ell^1 \cup Y_\ell^3 \cup Z_\ell^3$	$(u_\ell^4, v_\ell^7)$		$X_\ell^2 \cup Y_\ell^3 \cup Z_\ell^3$	$(u_\ell^4, v_\ell^7)$		$X_\ell^3 \cup Y_\ell^3 \cup Z_\ell^3$	$(u_\ell^2, v_\ell^5)$

is a woman  $w_\ell \in \{v_\ell^1, v_\ell^2, v_\ell^3\}$  who is single in  $M$ . Then  $(m_\ell, w_\ell)$  blocks  $M$  in  $L_1$  and  $L_2$ , a contradiction. Therefore,  $u_\ell^1$ ,  $u_\ell^2$ , and  $u_\ell^3$  are matched with  $v_\ell^1$ ,  $v_\ell^2$ , and  $v_\ell^3$  in  $M$ . There are six possible ways, namely,

$$\begin{aligned} X_\ell^1 &= \{(u_\ell^1, v_\ell^1), (u_\ell^2, v_\ell^2), (u_\ell^3, v_\ell^3)\}, X_\ell^2 = \{(u_\ell^1, v_\ell^2), (u_\ell^2, v_\ell^3), (u_\ell^3, v_\ell^1)\}, \\ X_\ell^3 &= \{(u_\ell^1, v_\ell^3), (u_\ell^2, v_\ell^1), (u_\ell^3, v_\ell^2)\}, X_\ell^4 = \{(u_\ell^1, v_\ell^1), (u_\ell^2, v_\ell^3), (u_\ell^3, v_\ell^2)\}, \\ X_\ell^5 &= \{(u_\ell^1, v_\ell^2), (u_\ell^2, v_\ell^1), (u_\ell^3, v_\ell^3)\}, \text{ and } X_\ell^6 = \{(u_\ell^1, v_\ell^3), (u_\ell^2, v_\ell^2), (u_\ell^3, v_\ell^1)\}. \end{aligned}$$

It is easy to see that  $X_\ell^4$  is blocked by  $(u_\ell^3, v_\ell^1)$ ,  $X_\ell^5$  is blocked by  $(u_\ell^2, v_\ell^3)$ , and  $X_\ell^6$  is blocked by  $(u_\ell^1, v_\ell^2)$  in  $L_1$ . Therefore, only  $X_\ell^1$ ,  $X_\ell^2$ , and  $X_\ell^3$  can be a part of  $M$ . The same argument applies to  $u_\ell^4$ ,  $u_\ell^5$ ,  $u_\ell^6$ ,  $v_\ell^4$ ,  $v_\ell^5$ ,  $v_\ell^6$  and  $u_\ell^7$ ,  $u_\ell^8$ ,  $u_\ell^9$ ,  $v_\ell^7$ ,  $v_\ell^8$ ,  $v_\ell^9$ , implying that only

$$Y_\ell^1 = \{(u_\ell^4, v_\ell^4), (u_\ell^5, v_\ell^5), (u_\ell^6, v_\ell^6)\}, Y_\ell^2 = \{(u_\ell^4, v_\ell^5), (u_\ell^5, v_\ell^6), (u_\ell^6, v_\ell^4)\},$$

$$Y_\ell^3 = \{(u_\ell^4, v_\ell^6), (u_\ell^5, v_\ell^4), (u_\ell^6, v_\ell^5)\},$$

and

$$Z_\ell^1 = \{(u_\ell^7, v_\ell^7), (u_\ell^8, v_\ell^8), (u_\ell^9, v_\ell^9)\}, Z_\ell^2 = \{(u_\ell^7, v_\ell^8), (u_\ell^8, v_\ell^9), (u_\ell^9, v_\ell^7)\},$$

$$Z_\ell^3 = \{(u_\ell^7, v_\ell^9), (u_\ell^8, v_\ell^7), (u_\ell^9, v_\ell^8)\}$$

are valid.

Therefore, there are 27 possible combinations. Note that  $M_\ell^1 = X_\ell^3 \cup Y_\ell^1 \cup Z_\ell^2$ ,  $M_\ell^2 = X_\ell^2 \cup Y_\ell^3 \cup Z_\ell^1$ , and  $M_\ell^3 = X_\ell^1 \cup Y_\ell^2 \cup Z_\ell^3$ . We show that the remaining 24 matchings are unstable in  $L_2$ . Table 1 shows 27 matchings in “Matching” columns and corresponding blocking pairs of 24 matchings in “BP” columns. This completes the proof. ◀

By Lemma 3, either  $M_{i,j}^1 \subseteq M$  for all  $j$  or  $M_{i,j}^0 \subseteq M$  for all  $j$  holds. In the former case, we set  $T(x_i) = 1$ , otherwise, we set  $T(x_i) = 0$ . We show that  $T$  satisfies  $f$ . Suppose not, and let  $C_\ell$  be an unsatisfied clause. For  $t = 1, 2, 3$ , let the  $t$ th literal of  $C_\ell$  be the  $j_t$ th occurrence of the variable  $x_{i_t}$ . We will show three claims:

**Claim 1.**  $M_\ell^1 \not\subseteq M$ . Consider the first literal of  $C_\ell$ . Suppose that it appears positively in  $C_\ell$ . Then by construction of the preference lists, the lists of  $b_{i_1,j_1}$  and  $v_\ell^4$  in  $L_1$  are as follows:

$$b_{i_1,j_1} : d_{i_1,j_1} \ v_\ell^4 \ c_{i_1,j_1} \quad v_\ell^4 : u_\ell^5 \ u_\ell^6 \ b_{i_1,j_1} \ u_\ell^4$$

Since  $C_\ell$  is unsatisfied,  $T(x_{i_1}) = 0$  and so by construction of  $T$ ,  $M_{i_1,j_1}^0 \subseteq M$ , i.e.,  $M(b_{i_1,j_1}) = c_{i_1,j_1}$ . If  $M_\ell^1 \subseteq M$ , then  $M(v_\ell^4) = u_\ell^4$  and hence  $(b_{i_1,j_1}, v_\ell^4)$  blocks  $M$  in  $L_1$ , a contradiction.

Next, suppose that the first literal of  $C_\ell$  is negative, i.e.,  $\overline{x_{i_1}}$ . Then by construction, the preference lists of  $d_{i_1,j_1}$  and  $u_\ell^1$  in  $L_1$  are as follows:

$$u_\ell^1 : v_\ell^1 \ v_\ell^2 \ d_{i_1,j_1} \ v_\ell^3 \quad d_{i_1,j_1} : a_{i_1,j_1} \ u_\ell^1 \ b_{i_1,j_1}$$

Since  $C_\ell$  is unsatisfied,  $T(x_{i_1}) = 1$  and so by construction of  $T$ ,  $M_{i_1,j_1}^1 \subseteq M$ , i.e.,  $M(d_{i_1,j_1}) = b_{i_1,j_1}$ . If  $M_\ell^1 \subseteq M$ , then  $M(u_\ell^1) = v_\ell^3$  and hence  $(u_\ell^1, d_{i_1,j_1})$  blocks  $M$  in  $L_1$ , a contradiction. Therefore, we can conclude that  $M_\ell^1 \not\subseteq M$ .

**Claim 2.**  $M_\ell^2 \not\subseteq M$ . Consider the second literal of  $C_\ell$ , and first suppose that it is a positive literal, i.e.,  $x_{i_2}$ . Then by construction, the preference lists of  $b_{i_2,j_2}$  and  $v_\ell^7$  in  $L_1$  are as follows:

$$b_{i_2,j_2} : d_{i_2,j_2} \ v_\ell^7 \ c_{i_2,j_2} \quad v_\ell^7 : u_\ell^8 \ u_\ell^9 \ b_{i_2,j_2} \ u_\ell^7$$

Since  $C_\ell$  is unsatisfied,  $T(x_{i_2}) = 0$  and hence by construction of  $T$ ,  $M_{i_2,j_2}^0 \subseteq M$ , i.e.,  $M(b_{i_2,j_2}) = c_{i_2,j_2}$ . If  $M_\ell^2 \subseteq M$ , then  $M(v_\ell^7) = u_\ell^7$  and hence  $(b_{i_2,j_2}, v_\ell^7)$  blocks  $M$  in  $L_1$ , a contradiction.

Next, suppose that the second literal of  $C_\ell$  is  $\overline{x_{i_2}}$ . Then by construction, the preference lists of  $d_{i_2,j_2}$  and  $u_\ell^4$  in  $L_1$  are as follows:

$$u_\ell^4 : v_\ell^4 \ v_\ell^5 \ d_{i_2,j_2} \ v_\ell^6 \quad d_{i_2,j_2} : a_{i_2,j_2} \ u_\ell^4 \ b_{i_2,j_2}$$

Since  $C_\ell$  is unsatisfied,  $T(x_{i_2}) = 1$  and by construction of  $T$ ,  $M_{i_2,j_2}^1 \subseteq M$ , i.e.,  $M(d_{i_2,j_2}) = b_{i_2,j_2}$ . If  $M_\ell^2 \subseteq M$ , then  $M(u_\ell^4) = v_\ell^6$  and hence  $(u_\ell^4, d_{i_2,j_2})$  blocks  $M$  in  $L_1$ , a contradiction. Therefore, we can conclude that  $M_\ell^2 \not\subseteq M$ .

**Claim 3.**  $M_\ell^3 \not\subseteq M$ . Consider the third literal of  $C_\ell$ . First, suppose that it is a positive literal  $x_{i_3}$ . Then by construction, the preference lists of  $b_{i_3,j_3}$  and  $v_\ell^1$  in  $L_1$  are as follows:

$$b_{i_3,j_3} : d_{i_3,j_3} \ v_\ell^1 \ c_{i_3,j_3} \quad v_\ell^1 : u_\ell^2 \ u_\ell^3 \ b_{i_3,j_3} \ u_\ell^1$$

Since  $C_\ell$  is unsatisfied,  $T(x_{i_3}) = 0$  and thus by construction of  $T$ ,  $M_{i_3,j_3}^0 \subseteq M$ , i.e.,  $M(b_{i_3,j_3}) = c_{i_3,j_3}$ . If  $M_\ell^3 \subseteq M$ , then  $M(v_\ell^1) = u_\ell^1$  and hence  $(b_{i_3,j_3}, v_\ell^1)$  blocks  $M$  in  $L_1$ , a contradiction.

Next, suppose that the third literal of  $C_\ell$  is negative, i.e.,  $\overline{x_{i_3}}$ . Then by construction, the preference lists of  $d_{i_3,j_3}$  and  $u_\ell^7$  in  $L_1$  are as follows:

$$u_\ell^7 : v_\ell^7 \ v_\ell^8 \ d_{i_3,j_3} \ v_\ell^9 \quad d_{i_3,j_3} : a_{i_3,j_3} \ u_\ell^7 \ b_{i_3,j_3}$$

Since  $C_\ell$  is unsatisfied,  $T(x_{i_3}) = 1$  and by construction of  $T$ ,  $M_{i_3,j_3}^1 \subseteq M$ , i.e.,  $M(d_{i_3,j_3}) = b_{i_3,j_3}$ . If  $M_\ell^3 \subseteq M$ , then  $M(u_\ell^7) = v_\ell^9$  and hence  $(u_\ell^7, d_{i_3,j_3})$  blocks  $M$  in  $L_1$ , a contradiction. Therefore, we can conclude that  $M_\ell^3 \not\subseteq M$ .

From Claims 1, 2, and 3, none of  $M_\ell^1$ ,  $M_\ell^2$ , and  $M_\ell^3$  can be a part of  $M$ , but this contradicts Lemma 4. Hence we conclude that  $T$  satisfies  $f$ , which completes the proof of Theorem 1.  $\blacktriangleleft$



In the above reduction, we have exploited existence of pairs that are acceptable in  $L_1$  but not in  $L_2$ , or vice versa. Then one may be curious about whether  $\text{SM}k\text{I}$  is solvable in polynomial time if the set of acceptable pairs is the same in all  $L_i$ . However, this is unlikely, as shown in the following corollary. Let  $\text{SM}k$  denote the special case of  $\text{SM}k\text{I}$  where all the preference lists are complete. Clearly  $\text{SM}k$  satisfies the above mentioned condition.

► **Corollary 5.** *For  $k \geq 2$ ,  $\text{SM}k$  is NP-complete.*

**Proof.** Apparently  $\text{SM}k \in \text{NP}$ . For the NP-hardness, in the reduction given in the proof of Theorem 1, make every preference list complete by appending missing persons to the tail of the list in an arbitrary order. It is not hard to see that the same correctness proof (with slight modifications) applies. ◀

### 3 Tractable Cases

In this section, we assume without loss of generality that acceptability is mutual, i.e.,  $m$  is acceptable to  $w$  in  $L_i$  if and only if  $w$  is acceptable to  $m$  in  $L_i$ . This is because, if for example  $m$  is acceptable to  $w$  while  $w$  is not acceptable to  $m$ , then  $(m, w)$  can neither be a part of a matching nor a blocking pair. Hence we may remove  $m$  from  $w$ 's list safely, without changing the set of jointly stable matchings. This preprocessing can be done in time linear in the total length of the input preference lists.

However, even if  $(m, w)$  is an acceptable pair in  $L_i$  but is an unacceptable pair in  $L_j$  ( $j \neq i$ ), we must not remove  $m$  and  $w$  from each other's list in  $L_i$ . This is because, although  $(m, w)$  cannot be a pair in a jointly stable matching, it may block some matching in  $L_i$  and removing it may change the set of jointly stable matchings.

#### 3.1 Length–Two Preferences Lists of One Side

Our first positive result is for instances in which the length of preference lists of one side, say men's side, is bounded by two. The proof of Theorem 6 exploits a partially-ordered set (poset) of rotations and its relation to the whole set of stable matchings. These structural properties were originally studied for complete preference lists, but they can be extended easily and naturally to incomplete preference lists. Here we give brief explanations about them. See [5] for more detail. Readers who are familiar with these notions may skip the following two paragraphs.

Let  $I$  be an instance of  $\text{SMI}$  and  $M$  be a stable matching for  $I$ . For a man  $m$  matched in  $M$ ,  $s_M(m)$  denotes the first woman  $w$  in  $m$ 's list such that  $w$  is matched in  $M$  and  $w$  prefers  $m$  to  $M(w)$ . Note that  $m$  prefers  $M(m)$  to  $s_M(m)$ ; otherwise,  $(m, s_M(m))$  blocks  $M$ . Also,  $\text{next}_M(m)$  denotes the partner of  $s_M(m)$  in  $M$ , that is,  $\text{next}_M(m) = M(s_M(m))$ . Let  $\rho = (m_0, w_0), (m_1, w_1), \dots, (m_{r-1}, w_{r-1})$  ( $r \geq 2$ ) be a sequence of pairs such that each pair in  $\rho$  is contained in  $M$  and  $m_{i+1} = \text{next}_M(m_i)$  for each  $i$ , where  $i + 1$  is taken modulo  $r$ . Then we call  $\rho$  a *rotation exposed in  $M$* . By *eliminating* a rotation  $\rho$  from  $M$ , we mean to replace pairs  $(m_0, w_0), (m_1, w_1), \dots, (m_{r-1}, w_{r-1})$  by  $(m_0, w_1), (m_1, w_2), \dots, (m_{r-1}, w_0)$  in  $M$ . The resulting matching, denoted by  $M/\rho$ , is also stable in  $I$ . Note that each man included in  $\rho$  has a worse partner in  $M/\rho$  than in  $M$ .

Let  $\Pi$  be the set of rotations that are exposed in one or more stable matchings for  $I$ . We can define a partial order  $\preceq$  on  $\Pi$ , and  $(\Pi, \preceq)$  is called the *rotation poset* of  $I$ . A subset  $P \subseteq \Pi$  is called a *closed subset* of  $\Pi$  if  $\rho \in P$  and  $\rho' \preceq \rho$  then  $\rho' \in P$ . There is a one-to-one correspondence between the stable matchings for  $I$  and the closed subsets of  $\Pi$  by the mapping defined as follows. Let  $M_0$  be the *man-optimal stable matching* of  $I$  (which is

guaranteed to exist and can be found by the *men-oriented Gale-Shapley algorithm* in time linear in the total length of preference lists). Let  $P$  be a closed subset of  $\Pi$ . If we eliminate rotations in  $P$  one by one according to the order  $\preceq$ , we obtain a stable matching for  $I$ . Conversely, any stable matching for  $I$  is obtained by this procedure for some closed subset of  $\Pi$ . In particular, the empty set corresponds to the man-optimal stable matching and the whole set  $\Pi$  corresponds to the *woman-optimal stable matching* (which is the opposite extreme to the man-optimal stable matching). The rotation poset can be constructed in time linear in the total length of preference lists (Sec. 3.3 of [5]).

► **Theorem 6.**  $(2, \infty)$ -SMkI is solvable in time  $O(kn)$ .

**Proof.** We first compute the man-optimal stable matchings  $M_i$  for  $L_i$  ( $i = 1, 2, \dots, k$ ) using the men-oriented version of the Gale-Shapley algorithm. For each  $L_i$ , any stable matching leaves the same set of men and women unmatched [4]. Thus if there are  $i$  and  $j$  ( $i \neq j$ ) such that the set of matched people in  $M_i$  and that in  $M_j$  are different, then we can immediately answer “no”. In the following, we assume that the sets of matched people are the same in all  $M_i$ .

For each  $i$ , we compute all the rotations  $\rho_1^i, \rho_2^i, \dots, \rho_{n_i}^i$  with respect to  $L_i$ . Since the length of each man’s preference list is at most two, each man is contained in at most one rotation. This means that all the rotations are mutually incomparable in the rotation poset. Hence there is a one-to-one correspondence between the set of stable matchings for  $L_i$  and the power set of  $\{\rho_1^i, \rho_2^i, \dots, \rho_{n_i}^i\}$ : the subset  $S \subseteq \{\rho_1^i, \rho_2^i, \dots, \rho_{n_i}^i\}$  corresponds to the stable matching  $M_{i,S}$  obtained by eliminating all the rotations in  $S$  from  $M_i$ . Consider a man  $m$  who is matched in  $M_i$ . If  $m$  is not included in a rotation, his partner is the same in all the stable matchings of  $L_i$ . If he is included in a rotation  $\rho_j^i$ , then he is matched in  $M_{i,S}$  with his first choice if  $\rho_j^i \notin S$  and with his second choice if  $\rho_j^i \in S$ .

The remaining task is to check if there are  $k$  subsets  $S_i \subseteq \{\rho_1^i, \rho_2^i, \dots, \rho_{n_i}^i\}$  ( $1 \leq i \leq k$ ) such that  $M_{1,S_1} = M_{2,S_2} = \dots = M_{k,S_k}$ . For this purpose, we introduce a binary variable  $x_j^i$  for  $\rho_j^i$  ( $1 \leq i \leq k, 1 \leq j \leq n_i$ ), where  $x_j^i = 1$  means to put  $\rho_j^i$  in  $S_i$ . We then construct a 2CNF SAT instance as follows.

For each man  $m$  who is matched in  $M_1$  (and equivalently in all  $M_i$ ), we fix the value of variables or construct 2CNF clauses to ensure that  $m$ ’s partners coincide in all  $M_{1,S_1}, M_{2,S_2}, \dots, M_{k,S_k}$ . If  $(m, w)$  is a pair in some stable matching of  $L$ ,  $w$  is called *m’s stable partner in  $L$* . Also, if  $w$  is  $m$ ’s stable partner in all  $L_i$ ,  $w$  is called *m’s jointly stable partner*. If  $m$  has no jointly stable partner, we immediately output “no”. If  $m$  has one jointly stable partner  $w$ , then for each  $i$ , we enforce the variable (if any) to match  $m$  with  $w$  in  $M_{i,S_i}$ . Namely, if  $m$  is not included in a rotation, then there is no variable and we do nothing. If  $m$  is included in a rotation  $\rho_j^i$  and  $w$  is his first (second) choice in  $L_i$ , then we set  $x_j^i = 0$  ( $x_j^i = 1$ ). During this course, if some variable is fixed differently, then we immediately output “no”. Finally, suppose that  $m$  has two jointly stable partners  $w'$  and  $w''$ . This means that for each  $i$ ,  $L_i(m)$  contains both  $w'$  and  $w''$  and  $m$  is included in a rotation of  $L_i$ . Let  $\rho_{j_i}^i$  be the rotation that includes  $m$ . For  $i = 2, \dots, k$ , we construct two clauses as follows: If the order of  $w'$  and  $w''$  is same in  $L_1(m)$  and  $L_i(m)$ , then we construct  $(x_{j_1}^1 \vee \overline{x_{j_i}^i})$  and  $(\overline{x_{j_1}^1} \vee x_{j_i}^i)$ ; otherwise, we construct  $(x_{j_1}^1 \vee x_{j_i}^i)$  and  $(\overline{x_{j_1}^1} \vee \overline{x_{j_i}^i})$ . The construction of 2CNF formula is completed by doing this for all the men  $m$  who are matched in  $M_1$ . It is not hard to see that a satisfying assignment corresponds to subsets  $S_i$  such that  $M_{1,S_1} = M_{2,S_2} = \dots = M_{k,S_k}$ .

Recall that men’s preference lists are of length at most two and acceptability is mutual by assumption, so the total lengths of  $L_i$  is  $O(n)$ . Therefore, for each  $i$ , finding  $M_i$  and computing the set of rotations of  $L_i$  can be done in  $O(n)$  time, and hence in  $O(kn)$  time

in total. Constructing 2CNF clauses for each man can be done in time  $O(k)$ , and therefore  $O(kn)$  for at most  $n$  men. The resulting 2CNF formula has size  $O(kn)$ . Finally, solving 2CNF satisfiability problem can be done in linear time [2, 1]. Thus overall time-complexity is  $O(kn)$ . ◀

### 3.2 Identical Preference Lists of One Side

The next polynomial-time solvable case is that each woman's preference lists are identical in all  $L_i$ . It should be noted that this condition is different from the so-called *master lists*, in which all the women have the same preference list. In our case,  $w$  and  $w'$  may have different preference lists.

► **Theorem 7.** *If each woman's preference lists in all  $L_i$  ( $1 \leq i \leq k$ ) are identical, SMkI is solvable in time  $O(N)$ , where  $N$  is the total length of preference lists in an input.*

**Proof.** Let  $I = (U, W, L_1, L_2, \dots, L_k)$  be an instance of SMkI. We first note that, since  $L_1(w) = L_2(w) = \dots = L_k(w)$  for every woman  $w$ , for each man  $m$  the sets of women included in  $L_i(m)$  are the same for all  $i$ , due to the mutual-acceptability assumption made at the beginning of this section. Now we construct a set  $L$  of preference lists from  $L_1, L_2, \dots, L_k$  as follows: For each woman  $w$ , let  $L(w) := L_1(w)$ . For each man  $m$ , the set of women included in  $L(m)$  is the same as in  $L_i(m)$ , and their order is defined as follows. Let  $w'$  and  $w''$  be women in  $L(m)$ . If  $m$  prefers  $w'$  to  $w''$  in all  $L_i(m)$ , then  $m$  prefers  $w'$  to  $w''$  in  $L(m)$ . If  $m$  prefers  $w'$  to  $w''$  in some  $L_i(m)$  and  $w''$  to  $w'$  in some  $L_j(m)$ , then  $m$  is indifferent between  $w'$  and  $w''$  in  $L(m)$ . It is not hard to see that  $L(m)$  is a partially-ordered list and hence  $I' = (U, W, L)$  can be regarded as an instance of the *Stable Marriage problem with Partially-ordered and Incomplete lists* (SMPI).

We now recall the *super-stability* [5, 6] in the case that preference lists are not necessarily in a total order. For a matching  $M$ ,  $(m, w)$  is a *blocking pair in super-stability* if (1)  $(m, w) \notin M$  but  $m$  and  $w$  are acceptable to each other, (2)  $m$  is single in  $M$ , or prefers  $w$  to  $M(m)$ , or is indifferent between  $w$  and  $M(m)$ , and (3)  $w$  is single in  $M$ , or prefers  $m$  to  $M(w)$ , or is indifferent between  $m$  and  $M(w)$ . We say that a matching is *super-stable* if it admits no blocking pair in super-stability. Irving [6] developed an  $O(n^2)$ -time algorithm to find a super-stable matching or to report that no super-stable matching exists when preference lists are complete and may include ties. Manlove [8] extended this algorithm for incomplete preference lists, and showed that it runs in time  $O(N)$  where  $N$  is the total length of preference lists in an input. Also, Manlove showed that the same algorithm is applicable for partially-ordered preference lists, i.e., SMPI (page 169 of [10]). Therefore, to complete the proof, it suffices to show that a matching  $M$  is jointly stable in  $I$  if and only if  $M$  is super-stable in  $I'$ .

First suppose that  $M$  is not a jointly stable matching of  $I$  and hence has a blocking pair  $(m, w)$  in  $L_i$  for some  $i$ . Then  $w$  is single in  $M$  or prefers  $m$  to  $M(w)$  in  $L_i(w)$ . In the latter case,  $w$  prefers  $m$  to  $M(w)$  also in  $L(w)$ . Similarly,  $m$  is single in  $M$  or prefers  $w$  to  $M(m)$  in  $L_i(m)$ . In the latter case,  $m$  prefers  $w$  to  $M(m)$  or is indifferent between them in  $L(m)$ . Thus  $(m, w)$  is a blocking pair in super-stability for  $M$  and therefore  $M$  is not super-stable in  $I'$ .

Conversely, suppose that  $M$  is not super-stable in  $I'$ . Then, there is a blocking pair  $(m, w)$  in super-stability. Since  $L(w)$  is a total order,  $w$  is unmatched in  $M$  or prefers  $m$  to  $M(w)$  in  $L(w)$ . In the latter case,  $w$  prefers  $m$  to  $M(w)$  in all  $L_i(w)$ . Note that  $m$  either (i) is unmatched in  $M$ , or (ii) prefers  $w$  to  $M(m)$  in  $L(m)$ , or (iii) is indifferent between  $w$  and  $M(m)$  in  $L(m)$ . In the case of (i),  $(m, w)$  is a blocking pair for  $M$  in all  $L_i$ . In the case of

(ii),  $m$  prefers  $w$  to  $M(m)$  in all  $L_i(m)$ , so again  $(m, w)$  is a blocking pair for  $M$  in all  $L_i$ . In the case of (iii),  $m$  prefers  $w$  to  $M(m)$  in  $L_i(m)$  for some  $i$ , so that  $(m, w)$  is a blocking pair for  $M$  in  $L_i$ . In any case,  $M$  is not jointly stable in  $I$ .

Constructing  $I'$  from  $I$  and solving  $I'$  can both be done in  $O(N)$  time, hence the theorem follows.  $\blacktriangleleft$

As a byproduct of the above proof, we can show the existence of the man-optimal and woman-optimal stable matchings. Let us call a jointly stable matching  $M$  *man-optimal* if for any man  $m$  and any jointly stable matching  $M'$ , either  $M(m) = M'(m)$  or  $m$  prefers  $M(m)$  to  $M'(m)$  in all  $L_i$ . The *woman-optimal* jointly stable matching is defined similarly.

Let  $I = (U, W, L_1, L_2, \dots, L_k)$  be an SMkI instance and  $I' = (U, W, L)$  be the SMPI instance constructed as in the above proof. It is known that the set of super-stable matchings for an SMPI instance form a distributive lattice ([12, 9] and page 169 of [10]), so there are the man-optimal and the woman-optimal stable matchings for  $I'$ , denoted  $M_U$  and  $M_W$ , respectively. Since women's preference lists are the same in  $L$  and all  $L_i$ ,  $M_W$  is the woman-optimal jointly stable matching for  $I$ . Consider a man  $m$  and suppose that  $m$  is indifferent between  $w_1$  and  $w_2$  in  $L(m)$ . It is known that it cannot be the case that  $m$  is matched with  $w_1$  in one super-stable matching and with  $w_2$  in another super-stable matching. Thus by the man-optimality of  $M_U$ , for every man  $m$ , either  $M_U(m) = M(m)$  or  $m$  prefers  $M_U(m)$  to  $M(m)$  in  $L(m)$  for any super-stable matching  $M$ . This implies that by construction of  $L$ , either  $M_U(m) = M(m)$  or  $m$  prefers  $M_U(m)$  to  $M(m)$  in  $L_i(m)$  for all  $i$ , implying the existence of the man-optimal jointly stable matching.

## 4 Conclusion

In this paper, we considered a variant of the stable marriage problem in which we are given  $k$  sets of preference lists  $L_1, L_2, \dots, L_k$ , and are asked to determine the existence of a matching that is stable with respect to all  $L_i$  ( $1 \leq i \leq k$ ). We have shown that the problem is NP-complete for  $k \geq 2$  even if all the preference lists are of length at most four, while it is solvable in linear time if each man's preference list is of length at most two. We also showed that the problem is solvable in linear time if every woman has an identical preference list in all  $L_i$ .

An important future work is to determine the complexity of the problem when the lengths of preference lists are bounded by three, namely,  $(3, \ell)$ -SMkI for  $\ell \geq 3$ . Another direction is approximability of SMkI; given an instance, find a matching that is stable in as many  $L_i$  as possible. Finding a stable matching in any one list is a trivial  $k$ -approximation algorithm. On the other hand, using Theorem 1 we can easily deduce an approximation hardness of  $2 - \epsilon$  for even  $k$  and  $2 - \frac{2}{k+1} - \epsilon$  for odd  $k$ , for any positive constant  $\epsilon$  under  $P \neq NP$ . Narrowing this gap is an interesting future work. Considering an alternative optimization criteria, e.g., minimizing the total number of blocking pairs over all  $L_i$ , would also be attractive.

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